THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 12

§91) 2) Note that $z^2+1=0$ if and only if $z=\pm i$. By Cauchy's Residue Theorem, for $R$ large enough and $\rho$ small enough we have

$$\int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz + \int_\rho^R f(z)dz + \int_{-R}^{-\rho} f(z)dz = 2\pi i \operatorname{Res}_{z=i} f(z)$$

Note that

$$\operatorname{Res}_{z=i} f(z) = \operatorname{Res}_{z=i} \frac{e^{-\log z/2}/(z+i)}{z-i} = \frac{e^{-\log i/2}}{2i} = \frac{e^{-i\pi/4}}{2i} = \frac{-\sqrt{2}-\sqrt{2}i}{4}$$

On $C_R$, since

$$|f(z)| \le \frac{e^{-\ln R/2}}{R^2-1} = \frac{R^{-1/2}}{R^2-1},$$

we have

$$\left|\int_{C_R} f(z)dz\right| \le \frac{R^{-1/2}}{R^2-1} \times \pi R = \frac{R^{1/2}}{R^2-1} \to 0$$

as $R\to\infty$.

On the other hand, on $C_\rho$, similarly we have

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\rho^{-1/2}}{1-\rho^2} \times \pi\rho = \frac{\rho^{1/2}}{1-\rho^2} \to 0$$

as $\rho\to 0$.

Furthermore, we have

$$\int_{-R}^{-\rho} f(z)dz = \int_{-R}^{-\rho} \frac{e^{\frac{-1}{2}(\ln|x|+i\pi)}}{x^2+1}dx = -i\int_{-R}^{-\rho} \frac{e^{\frac{-1}{2}(\ln|x|)}}{x^2+1}dx = -i\int_{\rho}^{R} \frac{e^{\frac{-1}{2}(\ln|x|)}}{x^2+1}dx$$

As a result, by taking $R\to\infty$ and $\rho\to 0$, we get

$$(1-i)\int_0^\infty \frac{e^{\frac{-1}{2}(\ln|x|)}}{x^2+1}dx = \frac{1-i}{\sqrt{2}}\pi$$

Hence

$$\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}dx = \frac{\pi}{\sqrt{2}}$$

**Remark:** To find the upper bound for the function, usually we need to use the triangle inequality. For example, if $f(z)=\frac{1}{z+1}$, then for $R$ large enough and $\rho$ small enough we have

$$\left|\frac{1}{z+1}\right| \le \frac{1}{R-1} \text{ and } \left|\frac{1}{z+1}\right| \le \frac{1}{1-\rho}$$

You should be reminded that the upper bound should be non-negative. That leads to the different between these two inequalities.

§91) 4) Note that $(z+a)(z+b)=0$ if and only if $z=-a$ or $-b$. By Cauchy's Residue Theorem, for $\rho < b < a < R$ we have

$$\int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz + \int_\rho^R \frac{e^{\frac{1}{3}(\ln x)}}{(x+a)(x+b)}dx - \int_\rho^R \frac{e^{\frac{1}{3}(\ln x+2\pi i)}}{(x+a)(x+b)}dx$$
$$= 2\pi i\left(\operatorname{Res}_{z=-a} f(z) + \operatorname{Res}_{z=-b} f(z)\right)$$

Note that

$$\operatorname{Res}_{z=-a} f(z) = \operatorname{Res}_{z=-a} \frac{e^{\log z/3}/(z+b)}{z+a} = \frac{e^{\log(-a)/3}}{-a+b} = \frac{e^{\frac{1}{3}(\ln a+i\pi)}}{b-a} = \frac{a^{\frac{1}{3}}}{b-a}\cdot e^{\frac{i\pi}{3}}$$

$$\operatorname{Res}_{z=-b} f(z) = \operatorname{Res}_{z=-b} \frac{e^{\log z/3}/(z+a)}{z+b} = \frac{e^{\log(-b)/3}}{-b+a} = \frac{e^{\frac{1}{3}(\ln b+i\pi)}}{a-b} = \frac{b^{\frac{1}{3}}}{a-b}\cdot e^{\frac{i\pi}{3}}$$

On $C_R$, since

$$|f(z)| \le \frac{e^{\ln R/3}}{(R-a)(R-b)} = \frac{R^{-1/3}}{(R-a)(R-b)},$$

we have

$$\left|\int_{C_R} f(z)dz\right| \le \frac{R^{-1/3}}{(R-a)(R-b)} \times \pi R = \frac{R^{2/3}}{(R-a)(R-b)} \to 0$$

as $R\to\infty$.

On the other hand, on $C_\rho$, similarly we have

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\rho^{-1/3}}{(a-\rho)(b-\rho)} \times \pi\rho = \frac{\rho^{2/3}}{(a-\rho)(b-\rho)} \to 0$$

as $\rho\to 0$.

Furthermore, we have

$$\int_\rho^R \frac{e^{\frac{1}{3}(\ln x+2\pi i)}}{(x+a)(x+b)}dx = e^{\frac{2\pi i}{3}} \int_\rho^R \frac{e^{\frac{1}{3}(\ln x)}}{(x+a)(x+b)}dx$$

As a result, by taking $R\to\infty$ and $\rho\to 0$, we get

$$(1-e^{\frac{2\pi i}{3}})\int_\rho^R \frac{e^{\frac{1}{3}(\ln x)}}{(x+a)(x+b)}dx = 2\pi i\left(\frac{b^{\frac{1}{3}}-a^{\frac{1}{3}}}{a-b}\right)e^{\frac{i\pi}{3}}$$

Hence

$$\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{a^{\frac{1}{3}}-b^{\frac{1}{3}}}{a-b}$$

§92) 1) Note that

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_{|z|=1} \frac{1}{5+4\left(\frac{z+z^{-1}}{2i}\right)}\frac{dz}{iz} = \int_{|z|=1} \frac{1}{2z^2+5iz-2}dz$$

Moreover, $2z^2+5iz-2=0$ if and only if $z=-2i$ or $z=-\frac{i}{2}$.

Therefore, by Cauchy's Residue Theorem, we have

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = 2\pi i\operatorname{Res}_{z=-\frac{i}{2}} \frac{1}{2z^2+5iz-2} = 2\pi i\frac{1}{2(-\frac{i}{2}+2i)} = \frac{2\pi}{3}$$

§94) 1)

a) Since $f(z) = z^2$ has 2 zeros and 0 poles (counted with multiplicities) inside the contour $|z| = 1$, we have

$$\Delta_C \arg f(z) = 2\pi(2-0) = 4\pi$$

b) Since $f(z) = 1/z^2$ has 0 zeros and 2 poles (counted with multiplicities) inside the contour $|z| = 1$, we have

$$\Delta_C \arg f(z) = 2\pi(0-2) = -4\pi$$

c) Since $f(z) = (2z-1)^7/z^3$ has 7 zeros and 3 poles (counted with multiplicities) inside the contour $|z| = 1$, we have

$$\Delta_C \arg f(z) = 2\pi(7-3) = 8\pi$$

§94) 6)

a) Let $f(z) = -5z^4$ and $g(z) = z^6 + z^3 - 2z$. Note that on $|z| = 1$, we have

$$|g(z)| \leq |z|^6 + |z|^3 + 2|z| = 4 < 5 = |f(z)|$$

As a result, by Rouche's Theorem, the number of zeros of $f(z)+g(z)$ and $f(z)$ are the same. Since $f(z) = -5z^4$ has 4 zeros inside the contour $|z| = 1$, the number of zeros of $f(z)+g(z) = z^6 - 5z^4 + z^3 - 2z$ inside the contour $|z| = 1$ is 4.

b) Let $f(z) = 9$ and $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$. Note that on $|z| = 1$, we have

$$|g(z)| \leq 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8 < 9 = |f(z)|$$

As a result, by Rouche's Theorem, the number of zeros of $f(z)+g(z)$ and $f(z)$ are the same. Since $f(z) = 9$ has 0 zeros inside the contour $|z| = 1$, the number of zeros of $f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$ inside the contour $|z| = 1$ is 0.

c) Let $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$. Note that on $|z| = 1$, we have

$$|g(z)| \leq |z|^7 + |z| + 1 = 3 < 4 = |f(z)|$$

As a result, by Rouche's Theorem, the number of zeros of $f(z)+g(z)$ and $f(z)$ are the same. Since $f(z) = -4z^3$ has 3 zeros inside the contour $|z| = 1$, the number of zeros of $f(z)+g(z) = z^7 - 4z^3 + z - 1$ inside the contour $|z| = 1$ is 3.